

IMPRIMITIVITY FOR C^* -COACTIONS OF NON-AMENABLE GROUPS

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ABSTRACT. We give a condition on a full coaction (A, G, δ) of a (possibly) nonamenable group G and a closed normal subgroup N of G which ensures that Mansfield imprimitivity works; i.e. that $A \times_{\delta|} G/N$ is Morita equivalent to $A \times_{\delta} G \times_{\delta, r} N$. This condition obtains if N is amenable or δ is normal. It is preserved under Morita equivalence, inflation of coactions, the stabilization trick of Echterhoff and Raeburn, and on passing to twisted coactions.

1. INTRODUCTION

For a nondegenerate reduced coaction $\delta: A \rightarrow M(A \otimes C_r^*(G))$ and a closed normal amenable subgroup N of G , Mansfield [Man91, Theorem 27] showed that the iterated crossed product $A \times_{\delta} G \times_{\delta} N$ is Morita equivalent to the restricted crossed product $A \times_{\delta|} G/N$. Later, Phillips and Raeburn [PR89, Theorem 4.1] proved that for a closed normal amenable subgroup N of G and a twisted coaction of $(G, G/N)$ on A , there is an action $\tilde{\delta}$ of N on the twisted crossed product $A \times_{G/N} G$ such that $A \times_{G/N} G \times_{\tilde{\delta}} N$ is Morita equivalent to A ; this can be interpreted as a twisted version of Mansfield's theorem.

Both the above results are coaction versions of Green's imprimitivity theorem for actions, and hence play an important role in the study of induced representations of crossed products by coactions. Naturally, we would like to remove the amenability hypothesis on N , which is only there to ensure that the restricted coaction $\delta|: A \rightarrow M(A \otimes C_r^*(G/N))$ is well-defined.

Accordingly, in this paper we consider a nondegenerate full coaction $\delta: A \rightarrow M(A \otimes C^*(G))$ and a closed normal subgroup N of G . We also use the abstract characterization of the crossed product $A \times_{\delta} G$ using universal properties, whereas Mansfield represents the crossed product on Hilbert space. In this situation, the proof of Mansfield's

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theorem shows that $A \times G \times_r N$ is Morita equivalent to $\text{im}(j_A \times j_G|)$, where (j_A, j_G) is the canonical covariant homomorphism of (A, G, δ) into $M(A \times G)$. We conclude (Corollary 3.4) that if $j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$ is faithful, then there is an $A \times G \times_r N - A \times G/N$ imprimitivity bimodule $Y_{G/N}^G$, which we describe explicitly. When δ is nondegenerate and $j_A \times j_G|$ is faithful, we say *Mansfield imprimitivity works* for N and δ .

In Section 4 we show that Mansfield imprimitivity passes to twisted crossed products, extending the above-mentioned result of Phillips and Raeburn. More explicitly, if $(A, G, G/K, \delta, \tau)$ is a twisted coaction and N is a closed normal subgroup of G contained in K such that Mansfield imprimitivity works for N and δ , then a quotient of $Y_{G/N}^G$ is an $A \times_{G/K} G \times_r N - A \times_{G/K} G/N$ imprimitivity bimodule (Theorem 4.4).

We were led to these ideas by an investigation [KQR95] of the duality of induction and restriction for coactions, and hence by a need for a general, workable imprimitivity framework for Mansfield induction. In Section 5 we prove several results concerning the compatibility of Mansfield imprimitivity with coaction constructions such as Morita equivalence, inflation, and stabilization. These results will be needed in [KQR95]; for the present, they serve to illustrate the robustness of Mansfield imprimitivity.

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2. PRELIMINARIES

Throughout, G will be a locally compact group with modular function Δ_G and left Haar measure ds . The group C^* -algebra of G is denoted $C^*(G)$; a subscript r , as in $C_r^*(G)$ or $B \times_r G$, always indicates a reduced object.

Coactions. Our coactions use the conventions of [Qui95], [QR95], and [Rae92], although the latter uses maximal tensor products. A (full) *coaction* of G on A is an injective, nondegenerate homomorphism δ from A to $M(A \otimes C^*(G))$ (where we use the minimal tensor product for C^* -algebras throughout) such that:

- (i) $\delta(a)(1 \otimes z), (1 \otimes z)\delta(a) \in A \otimes C^*(G)$ for all $a \in A, z \in C^*(G)$;
- (ii) $(\delta \otimes \iota) \circ \delta = (\iota \otimes \delta_G) \circ \delta$,

where $\delta_G: C^*(G) \rightarrow M(C^*(G) \otimes C^*(G))$ is the integrated form of the representation $s \mapsto s \otimes s$ of G . The coaction δ is *nondegenerate* if $\overline{\delta_{A(G)}(A)} = A$, where $\delta_f(a) = (\iota \otimes f)(\delta(a))$ for f in the Fourier algebra $A(G)$ and a in A . Equivalently, δ is nondegenerate if $\overline{\delta(A)(1 \otimes C^*(G))} = A \otimes C^*(G)$ [Kat85, Theorem 5].

Suppose K is a closed normal subgroup of G , and (A, K, ϵ) is a coaction. Let $i_K: C^*(K) \rightarrow M(C^*(G))$ be the canonical nondegenerate homomorphism. Then by [PR94, Example 2.4],

$$\inf \epsilon = (\iota \otimes i_K) \circ \epsilon: A \rightarrow M(A \otimes C^*(G))$$

is a coaction of G on A , called the *inflation* of ϵ . It turns out that inflation respects nondegeneracy of coactions:

Proposition 2.1. *Let K be a closed normal subgroup of G , and let (A, K, ϵ) be a coaction. Then ϵ is nondegenerate if and only if the inflated coaction $(A, G, \inf \epsilon)$ is.*

Proof. Suppose first that ϵ is nondegenerate. Then

$$\begin{aligned} \overline{\inf \epsilon(A)(1 \otimes C^*(G))} &= \overline{(\iota \otimes i_K)(\epsilon(A))(1 \otimes i_K(C^*(K))C^*(G))} \\ &= \overline{(\iota \otimes i_K)(\epsilon(A)(1 \otimes C^*(K)))(1 \otimes C^*(G))} \\ &= \overline{(A \otimes i_K(C^*(K)))(1 \otimes C^*(G))} \\ &= A \otimes C^*(G), \end{aligned}$$

so $\inf \epsilon$ is nondegenerate.

Conversely, suppose $\inf \epsilon$ is nondegenerate. A simple calculation shows that $(\inf \epsilon)_f = \epsilon_{f|_K}$ for $f \in A(G)$. Now $\{f|_K \mid f \in A(G)\} \subset A(K)$, since $A(G)$ is the closure in $B(G)$ of $B(G) \cap C_c(G)$. Hence,

$$A = \overline{(\inf \epsilon)_{A(G)}(A)} \subset \overline{\epsilon_{A(K)}(A)},$$

so ϵ is nondegenerate. □

A *covariant representation* of a coaction (A, G, δ) is a pair (π, μ) , where π and μ are nondegenerate representations of A and $C_0(G)$, respectively, on Hilbert space, such that

$$(\pi \otimes \iota) \circ \delta(a) = \text{Ad } \mu \otimes \iota(w_G)(\pi(a) \otimes 1) \quad \text{for } a \in A,$$

where $w_G \in M(C_0(G) \otimes C^*(G))$ is the unitary element determined by the canonical embedding of G in $M(C^*(G))$. A *crossed product* for (A, G, δ) is a triple $(A \times_\delta G, j_A, j_G)$ such that $A \times_\delta G$ is a C^* -algebra (which we will denote by $A \times G$ if δ is understood) and (j_A, j_G) is a pair of nondegenerate homomorphisms of A and $C_0(G)$, respectively, to $M(A \times G)$ satisfying:

- (i) for every nondegenerate representation ρ of $A \times G$, $(\rho \circ j_A, \rho \circ j_G)$ is a covariant representation of (A, G, δ) ;
- (ii) for every covariant representation (π, μ) of (A, G, δ) there is a representation $\pi \times \mu$ of $A \times G$ such that $(\pi \times \mu) \circ j_A = \pi$ and $(\pi \times \mu) \circ j_G = \mu$;
- (iii) $A \times_\delta G$ is the closed span of the products $j_A(a)j_G(f)$ for $a \in A$, $f \in C_0(G)$.

We write j_A^G for j_A and j_G^A for j_G when confusion due to the presence of several coactions or groups is likely to arise. There is an action $\hat{\delta}$, called the *dual action*, of G on $A \times G$ such that $\hat{\delta}_s(j_A(a)j_G(f)) = j_A(a)j_G(s \cdot f)$, where $(s \cdot f)(t) = f(ts)$.

It turns out that, for any nondegenerate representation π of A , $((\pi \otimes \lambda) \circ \delta, 1 \otimes M)$, where λ is the left regular representation of G and M is the multiplication representation of $C_0(G)$ on $L^2(G)$, is a covariant representation of (A, G, δ) , which moreover can be taken to be (j_A, j_G) whenever $\ker \pi \subset \ker j_A$. We call δ *normal* if j_A is faithful. If (π, μ) is a covariant representation, then $\text{Ad } \mu \otimes \iota(w_G) \circ (\cdot \otimes 1)$ is a normal coaction on $\pi(A)$. In particular, $\text{Ad } j_G \otimes \iota(w_G) \circ (\cdot \otimes 1)$ is a normal coaction on $j_A(A)$ with (essentially) the same covariant representations and crossed product as δ , and we call this coaction on $j_A(A)$ the *normalization* of δ . If δ is a normal coaction, then $(\iota \otimes \lambda) \circ \delta$ is a reduced coaction on A , which is nondegenerate if and only if δ is. In any event, whether δ is normal or not, $(\iota \otimes \lambda) \circ \delta$ factors to give a reduced coaction on $A/\ker j_A$, called the *reduction* of δ . Moreover, every nondegenerate reduced coaction is the reduction of a unique normal coaction.

For example, $(C^*(G), G, \delta_G)$ is a coaction which is normal if and only if G is amenable. $(\mathcal{K}(L^2(G)), \lambda, M)$ is a crossed product, so $\lambda_s \mapsto \lambda_s \otimes s$ is the normalization and $\lambda_s \mapsto \lambda_s \otimes \lambda_s$ the reduction.

Hilbert modules. Our main references for Hilbert modules and Morita equivalence are [Lan95], [Rie74], and [Rie79]. All our Hilbert modules (except multiplier bimodules) will be full; i.e., the closed span of the inner products generates the C^* -algebra.

Definition 2.2. A *right-Hilbert $A - B$ bimodule* (a term coined by Bui in [Bui95]) is a (right) Hilbert B -module X together with a nondegenerate action of A by adjointable B -module maps (i.e., there is a homomorphism of A into $\mathcal{L}_B(X)$ such that $AX = X$).

If X is also a left Hilbert A -module (in the obvious sense) such that ${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$ for $x, y, z \in X$, then of course X is an $A - B$ imprimitivity bimodule. We write ${}_AX_B$ when we want to emphasize

(or merely indicate) the coefficient algebras. We denote the reverse bimodule by \tilde{X} , with elements \tilde{x} .

For this work (and in [KQR95]), we feel that right-Hilbert bimodules are the right objects to use. For example, in applications of the Rieffel induction process it is really right-Hilbert bimodules that are required. Routine calculations suffice to adapt most results about Hilbert modules or imprimitivity bimodules to the setting of right-Hilbert bimodules, and we will use such adapted results with only a reference to the original results.

Following [ER95], a *multiplier* $m = (m_A, m_B)$ of an $A - B$ imprimitivity bimodule X consists of an A -linear map $m_A: A \rightarrow X$ and a B -linear map $m_B: B \rightarrow X$ such that $m_A(a) \cdot b = a \cdot m_B(b)$ for $a \in A$ and $b \in B$. The *multiplier bimodule* $M(X)$ consists of all multipliers of X ; it is naturally a Hilbert $M(A) - M(B)$ bimodule, but is in general *not* full (cf. [ER95, §1]). Following [ER95, Definition 1.8] and [Ng95, Definition A1(b)], an *imprimitivity bimodule homomorphism* $\varphi = (\varphi_A, \varphi_X, \varphi_B): {}_A X_B \rightarrow M({}_C Y_D)$ consists of homomorphisms $\varphi_A: A \rightarrow M(C)$ and $\varphi_B: B \rightarrow M(D)$ and a linear map $\varphi_X: X \rightarrow M(Y)$ satisfying

- (i) $\varphi_A({}_A \langle x, y \rangle) = {}_{M(C)} \langle \varphi_X(x), \varphi_X(y) \rangle$;
- (ii) $\varphi_B(\langle x, y \rangle_B) = \langle \varphi_X(x), \varphi_X(y) \rangle_{M(D)}$;
- (iii) $\varphi_X(a \cdot x \cdot b) = \varphi_A(a) \cdot \varphi_X(x) \cdot \varphi_B(b)$,

for all $a \in A$, $x, y \in X$, and $b \in B$.

Following [Ng95, Definition 3.3] (see also [BS89], [Bui94], [ER]), a *coaction* δ of G on an imprimitivity bimodule ${}_A X_B$ is an imprimitivity bimodule homomorphism

$$\delta = (\delta_A, \delta_X, \delta_B): {}_A X_B \rightarrow M({}_{A \otimes C^*(G)}(X \otimes C^*(G))_{B \otimes C^*(G)})$$

such that (A, G, δ_A) and (B, G, δ_B) are C^* -coactions, and satisfying

$$(\delta_X \otimes \iota) \circ \delta_X = (\iota \otimes \delta_G) \circ \delta_X$$

and

$$\overline{\delta_X(X)(B \otimes C^*(G))} = X \otimes C^*(G).$$

When such a δ exists we say (A, G, δ_A) and (B, G, δ_B) are *Morita equivalent*, and we call (X, δ_X) a *Morita equivalence* of δ_A and δ_B . Note that we automatically have $\delta_X(x) \cdot (1_B \otimes z)$ and $(1_A \otimes z) \delta_X(x) \in X \otimes C^*(G)$ for $x \in X$, $z \in C^*(G)$.

It turns out that Morita equivalence preserves nondegeneracy of C^* -coactions:

Proposition 2.3. *Let (A, G, δ_A) and (B, G, δ_B) be Morita equivalent coactions. Then δ_A is a nondegenerate coaction if and only if δ_B is.*

Proof. Suppose (X, δ_X) is a Morita equivalence for the C^* -coactions (A, G, δ_A) and (B, G, δ_B) ; so $\delta = (\delta_A, \delta_X, \delta_B)$ is a coaction of G on ${}_A X_B$.

Assume δ_B is nondegenerate. Then $\overline{\delta_B(B)(1 \otimes C^*(G))} = B \otimes C^*(G)$, so we have

$$\begin{aligned} \overline{\delta_X(X)(1 \otimes C^*(G))} &= \overline{\delta_X(X \cdot B)(1 \otimes C^*(G))} \\ &= \overline{\delta_X(X)\delta_B(B)(1 \otimes C^*(G))} \\ &= \overline{\delta_X(X)(B \otimes C^*(G))}. \end{aligned}$$

Thus,

$$(2.1) \quad \overline{\delta_X(X)(1 \otimes C^*(G))} = X \otimes C^*(G).$$

We will need to slice $M(X \otimes C^*(G))$ into $M(X)$. For this we have found it most convenient to use the linking algebra

$$L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$$

of [BGR77], which by [ER95, Appendix] carries a coaction defined by

$$\delta_L = \begin{pmatrix} \delta_A & \delta_X \\ \delta_{\tilde{X}} & \delta_B \end{pmatrix}.$$

Let $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We regard A , B , and X as sitting inside L , so in particular $X = pLq$. [ER95, Proposition A.1] shows that

$$M(L) = \begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix},$$

and we have $M(X) = pM(L)q$. Moreover,

$$L \otimes C^*(G) \cong \begin{pmatrix} A \otimes C^*(G) & X \otimes C^*(G) \\ \tilde{X} \otimes C^*(G) & B \otimes C^*(G) \end{pmatrix},$$

and we blur the distinction between the two sides of this isomorphism. Hence, we have

$$X \otimes C^*(G) = (p \otimes 1)(L \otimes C^*(G))(q \otimes 1)$$

and

$$M(X \otimes C^*(G)) = (p \otimes 1)(M(L \otimes C^*(G)))(q \otimes 1).$$

Now let $f \in A(G)$. We can certainly use $\iota \otimes f$ to slice $M(L \otimes C^*(G))$ into $M(L)$, and it makes sense to restrict $\iota \otimes f$ to $M(X \otimes C^*(G))$. We have

$$\begin{aligned} (\iota \otimes f)(M(X \otimes C^*(G))) &= (\iota \otimes f)((p \otimes 1)M(L \otimes C^*(G))(q \otimes 1)) \\ &= p(\iota \otimes f)(M(L \otimes C^*(G)))q \\ &\subset pM(L)q \\ &= M(X), \end{aligned}$$

so we now know how to slice $M(X \otimes C^*(G))$ into $M(X)$.

As with C^* -coactions, we really need to slice the image of δ_X into X . Without loss of generality let $f = g \cdot d$, with $g \in A(G)$ and $d \in C^*(G)$. Then

$$\begin{aligned} (\iota \otimes f)(\delta_X(X)) &= (\iota \otimes g)((1 \otimes d)\delta_X(X)) \\ &\subset (\iota \otimes g)(X \otimes C^*(G)) \\ &= (\iota \otimes g)((p \otimes 1)(L \otimes C^*(G))(q \otimes 1)) \\ &= p(\iota \otimes g)(L \otimes C^*(G))q \\ &\subset pLq \\ &= X. \end{aligned}$$

Hence, X is an $A(G)$ -submodule of L , as desired.

For $x \in (\iota \otimes A_c(G))(\delta_X(X))$, $f \in A_c(G)$, and $d \in C^*(G)$, [Qui95, Equation (1.2)] tells us that

$$x \otimes u_G(f)d = \int \delta_X((\iota \otimes s \cdot f)(x))(1 \otimes sd) ds,$$

where $u_G: L^1(G) \rightarrow C^*(G)$ is the canonical embedding, $(s \cdot f)(t) = f(ts)$, and the integral is norm-convergent. Thus, the proof of [Qui95, Corollary 1.5] shows that (2.1) implies

$$(2.2) \quad \overline{(\iota \otimes A(G))(\delta_X(X))} = X.$$

Now, (2.2) is a symmetric condition which is actually equivalent to the asymmetric (2.1), so we can expect to derive a mirror-image of (2.1) from it. Indeed, a calculation similar to the proof of [Qui95, Equation (1.2)] shows that

$$x \otimes du_G(f) = \int (1 \otimes ds)\delta_X((\iota \otimes (f \cdot s)')(x)) ds,$$

where $(f \cdot s)'(t) = f(st)\Delta(t)$. Thus, the proof of [Qui95, Corollary 1.4] shows that (2.2) implies

$$(2.3) \quad \overline{(1 \otimes C^*(G))\delta_X(X)} = X \otimes C^*(G).$$

(Note that the proof of [Qui95, Corollary 1.4] did not need to use [Qui95, Equation (1.1)].) Using (2.3) in the $M(A \otimes C^*(G))$ -valued inner product on $X \otimes C^*(G)$ gives

$$\overline{(1 \otimes C^*(G))\delta_A(A)(1 \otimes C^*(G))} = A \otimes C^*(G),$$

which implies δ_A is nondegenerate by an argument similar to the proof of [Qui95, Corollary 1.5].

We have shown that nondegeneracy of δ_B implies nondegeneracy of δ_A ; by symmetry, this completes the proof. \square

3. MANSFIELD IMPRIMITIVITY

Let (A, G, δ) be a coaction. Further let N be a closed normal subgroup of G , $q_N: C^*(G) \rightarrow C^*(G/N)$ the canonical quotient homomorphism, and $p_N: C_0(G/N) \rightarrow M(C_0(G))$ the nondegenerate embedding $p_N(f)(s) = f(sN)$. (When confusion is unlikely, we suppress p_N and identify $C_0(G/N)$ with its image in $C_b(G)$.) Then $\delta| := (\iota \otimes q_N) \circ \delta$ is a coaction of G/N on A , called the *restriction* of δ to G/N . Only injectivity is nonobvious, but for this note that $(\delta \otimes \iota) \circ \delta| = (\iota \otimes \delta_G|) \circ \delta$, and $\delta_G|$ is injective since it has left inverse $\iota \otimes \pi_0$, where $\pi_0: G/N \rightarrow \{1\}$ is the trivial character. If μ is a homomorphism of $C_0(G)$, we write $\mu|$ for $\mu \circ p_N$ when confusion seems unlikely, and think of $\mu|$ as the restriction of μ to $C_0(G/N)$. An easy computation, using the identity $(\iota \otimes q_N)(w_G) = (p_N \otimes \iota)(w_{G/N})$, shows that if (π, μ) is a covariant representation of (A, G, δ) , then $(\pi, \mu|)$ is a covariant representation of $(A, G/N, \delta|)$. In particular, since $C_0(G/N)$ sits nondegenerately inside $M(C_0(G))$,

$$j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$$

is a nondegenerate homomorphism. However, $j_A \times j_G|$ will be unfaithful in general. For example, if δ is any nonnormal coaction, then $A \times G/G = A$ and $j_A \times j_G| = j_A$ is unfaithful.

Lemma 3.1. *$j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$ is faithful if and only if $\ker j_A^G \subset \ker j_A^{G/N}$, and in this case we actually have $\ker j_A^G = \ker j_A^{G/N}$.*

Proof. Since $(j_A^G, j_G|)$ is a covariant representation of $(A, G/N, \delta|)$, we always have $\ker j_A^{G/N} \subset \ker j_A^G$, giving the second statement.

The forward implication of the first statement is clear: if $j_A \times j_G|$ is faithful, certainly $\ker j_A^G \subset \ker j_A^{G/N}$. For the reverse implication, assume $\ker j_A^G \subset \ker j_A^{G/N}$. Since the dual action $\hat{\delta}$ leaves the image of $j_A \times j_G|$ invariant, and since the restriction $\hat{\delta}|_N$ is trivial on $\text{im } j_A \times j_G|$,

we get an action of G/N on $\text{im } j_A \times j_G|$ such that $j_A \times j_G|$ is G/N -equivariant. Since $\ker j_A^G \subset \ker j_A^{G/N}$, [Qui95, Proposition 3.9] shows $j_A \times j_G|$ is faithful. \square

There are certainly lots of situations where $j_A \times j_G|$ is faithful:

Lemma 3.2. *$j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$ is faithful if either δ is normal, in which case $\delta|$ is normal as well, or N is amenable.*

Proof. If δ is normal, then $\ker j_A^G = \{0\}$, so by the preceding lemma $\ker j_A^{G/N} = \{0\}$ as well, hence $j_A \times j_G|$ is faithful and $\delta|$ is normal.

On the other hand, if N is amenable, then $\ker \lambda_G \subset \ker \lambda_{G/N} \circ q_N$, so

$$\begin{aligned} \ker j_A^G &= \ker(\iota \otimes \lambda_G) \circ \delta \subset \ker(\iota \otimes \lambda_{G/N}) \circ (\iota \otimes q_N) \circ \delta \\ &= \ker(\iota \otimes \lambda_{G/N}) \circ \delta| = \ker j_A^{G/N}, \end{aligned}$$

so again using the preceding lemma, $j_A \times j_G|$ is faithful. \square

We now adapt Mansfield's imprimitivity to our setting. Let (A, G, δ) be a coaction and N a closed normal subgroup of G . Since (j_A, j_G) can be taken to be $((\iota \otimes \lambda) \circ \delta, 1 \otimes M)$, where A is faithfully and nondegenerately represented on a Hilbert space \mathcal{H} , all of Mansfield's computations with $A \times G$ on $\mathcal{H} \otimes L^2(G)$ can be carried out abstractly with (j_A, j_G) . In Mansfield's setting (with δ reduced and N amenable), $((\iota \otimes \lambda) \circ \delta) \times (1 \otimes M|)$ is a faithful representation of $A \times G/N$ [Man91, Proposition 7]. In our setting, this representation of $A \times G/N$ need not be faithful. However, all of Mansfield's computations are really done with the image of $A \times G/N$ in $B(\mathcal{H} \otimes L^2(G))$. In our abstract setting, we use $\text{im } j_A \times j_G|$ (where j_A means j_A^G by default), and again all of Mansfield's computations carry over.

Mansfield develops the bulk of his imprimitivity machinery in §3 of [Man91]; note that he does not require N to be amenable for this section. However, he implicitly requires δ to be nondegenerate (and this becomes explicit in his proof of [Man91, Theorem 12]), so we will require this also from now on.

Let $A_c(G) = A(G) \cap C_c(G)$, and for a compact subset E of G let $C_E(G) = \{f \in C(G) \mid \text{supp } f \subset E\}$. For fixed $u \in A_c(G)$ and compact $E \subset G$, say an element of $M(A \times G)$ is (u, E, N) if it is in the closed span of the products $j_A(\delta_u(a))j_G(\varphi(f))$ for $a \in A$, $f \in C_E(G)$, where $\delta_u = (\iota \otimes u) \circ \delta: A \rightarrow A$ and $\varphi: C_c(G) \rightarrow C_c(G/N)$ is the surjection

$$\varphi(f)(sN) = \int_N f(sn) \, dn.$$

For $N = \{e\}$ just say the element is (u, E) . Mansfield's computations show the set \mathcal{D}_N of elements of $M(A \times G)$ which are (u, E, N) for some $u \in A_c(G)$ and compact $E \subset G$ is a dense $*$ -subalgebra of $\text{im } j_A \times j_G$; write \mathcal{D} for $\mathcal{D}_{\{e\}}$. We emphasize that when $A \times G$ is represented on $\mathcal{H} \otimes L^2(G)$ via $(\iota \otimes \lambda) \circ \delta \times (1 \otimes M)$ we get exactly Mansfield's \mathcal{D} and \mathcal{D}_N . Mansfield's computations also show there is a linear map $\Psi: \mathcal{D} \rightarrow \mathcal{D}_N$ such that

$$\Psi(j_A(a)j_G(f)) = j_A(a)j_G(\varphi(f)) \quad \text{for } a \in \delta_{A_c(G)}(A), f \in C_c(G).$$

Moreover, for $x, y \in \mathcal{D}$ the maps $n \mapsto \hat{\delta}_n(x)y$ and $n \mapsto x\hat{\delta}_n(y)$ are in $C_c(N, \mathcal{D})$, and

$$\begin{aligned} \Psi(x)y &= \int_N \hat{\delta}_n(x)y \, dn \\ x\Psi(y) &= \int_N x\hat{\delta}_n(y) \, dn. \end{aligned}$$

\mathcal{D} becomes a (full) pre-Hilbert \mathcal{D}_N -module under right multiplication (which makes sense since $\mathcal{D}_N \subset M(A \times G)$) and inner product

$$\langle x, y \rangle_{\mathcal{D}_N} = \Psi(x^*y).$$

\mathcal{D} becomes a left $C_c(N, \mathcal{D})$ -module under the integrated form of the left \mathcal{D} multiplication and the N -action

$$n \cdot x = \Delta(n)^{\frac{1}{2}} \hat{\delta}_n(x) \quad \text{for } n \in N, x \in \mathcal{D}.$$

Let $Y_{G/N}^G$ denote the completion of the pre-Hilbert \mathcal{D}_N -module \mathcal{D} . Then $Y_{G/N}^G$ is a right-Hilbert $A \times G \times N - \text{im } j_A \times j_G$ bimodule. The computations of [Man91, proof of Proposition 26] show that the homomorphism of $A \times G \times N$ to $\mathcal{L}_{\text{im } j_A \times j_G}(Y_{G/N}^G)$ has image $\mathcal{K}(Y_{G/N}^G)$ and the same kernel as the regular representation $A \times G \times N \rightarrow A \times G \times_r N$. He accomplishes the latter by showing that there is a faithful representation of $\mathcal{K}(Y_{G/N}^G)$, induced from the identity representation of $\text{im } j_A \times j_G$ (when (j_A, j_G) is taken to be $((\iota \otimes \lambda) \circ \delta, 1 \times M)$), such that the composition with $A \times G \times N \rightarrow \mathcal{K}(Y_{G/N}^G)$ is equivalent to the regular representation. Therefore, in our setting Mansfield's imprimitivity theorem becomes:

Theorem 3.3. [Man91, Theorem 27] *If (A, G, δ) is a nondegenerate coaction and N is a closed normal subgroup of G , then $Y_{G/N}^G$ is an $A \times G \times_r N - \text{im } j_A \times j_G$ imprimitivity bimodule.*

We would really like a Morita equivalence between $A \times G \times_r N$ and $A \times G/N$ itself, so clearly we need exactly the condition that $j_A \times j_G: A \times G/N \rightarrow M(A \times G)$ is faithful.

Corollary 3.4. *Let (A, G, δ) be a nondegenerate coaction and N a closed normal subgroup of G such that*

$$j_A \times j_G| : A \times G/N \rightarrow M(A \times G)$$

is faithful. Then $Y_{G/N}^G$ is an $A \times G \times_r N$ - $A \times G/N$ imprimitivity bimodule.

In view of Corollary 3.4, we make the following definition:

Definition 3.5. If (A, G, δ) is a nondegenerate coaction and N is a closed normal subgroup of G , we say *Mansfield imprimitivity works* for N and δ if $j_A \times j_G|$ is faithful.

When Mansfield imprimitivity works we let $\langle \cdot, \cdot \rangle_{A \times G/N}$ denote the extension to $Y_{G/N}^G$ of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}_N}$ on \mathcal{D} . Mansfield's computations show that the left inner product ${}_{A \times G \times_r N} \langle x, y \rangle$ for $x, y \in \mathcal{D}$ can be identified with the element

$${}_{A \times G \times_r N} \langle x, y \rangle(n) = x \hat{\delta}_n(y^*) \Delta(n)^{-\frac{1}{2}}$$

of $C_c(N, \mathcal{D})$.

The following lemma shows the strong connection between Mansfield imprimitivity working and the existence of a twist (see Section 4):

Lemma 3.6. *If $(A, G, G/K, \delta, \tau)$ is a nondegenerate twisted coaction, then Mansfield imprimitivity works for K and δ if and only if δ is normal.*

Proof. Since $(A, G/K, \delta|)$ is unitary, it is normal, so by definition

$$j_A^{G/K} : A \rightarrow M(A \times G/K)$$

is faithful. If Mansfield imprimitivity works for K , then

$$j_A^G \times j_G| : A \times G/K \rightarrow M(A \times G)$$

is faithful. Hence, the composition

$$j_A^G = (j_A^G \times j_G|) \circ j_A^{G/K}$$

is faithful as well, so δ is normal.

The converse is part of Lemma 3.2. □

To round out this discussion of Mansfield imprimitivity, we mention that we have been unable to find an example where δ is nonnormal and N is nonamenable, but Mansfield imprimitivity still works.

4. IMPRIMITIVITY FOR TWISTED COACTIONS

We now show how Mansfield imprimitivity passes to twisted crossed products, extending [PR94, Theorem 4.1]. If K is a closed normal subgroup of G , a coaction (A, G, δ) is *twisted* over G/K if there is a nondegenerate homomorphism $\tau: C_0(G/K) \rightarrow M(A)$, called the *twist*, such that:

- (i) $\delta| = \text{Ad } \tau \otimes \iota(w_{G/K}) \circ (\cdot \otimes 1)$;
- (ii) $\delta \circ \tau = \tau \otimes 1$;

alternatively we call $(A, G, G/K, \delta, \tau)$ a *twisted coaction*. A twisted coaction $(A, G, G/K, \delta, \tau)$ is *nondegenerate* or *normal* if the untwisted coaction (A, G, δ) is. A covariant representation (π, μ) of (A, G, δ) *preserves the twist* if $\pi \circ \tau = \mu|$. The *twisting ideal* of $A \times G$ is

$$I_\tau = \bigcap \{ \ker \pi \times \mu \mid (\pi, \mu) \text{ is a covariant representation of } (A, G, \delta) \text{ preserving the twist} \}.$$

The *twisted crossed product* is

$$A \times_{G/K} G = (A \times G) / I_\tau.$$

The quotient map $A \times G \rightarrow A \times_{G/K} G$ is of the form $k_A \times k_G$ for a unique covariant representation (k_A, k_G) . Moreover, (k_A, k_G) preserves the twist, and for any covariant representation (π, μ) preserving the twist there is a unique representation $\pi \times_{G/K} \mu$ of $A \times_{G/K} G$ such that

$$(\pi \times_{G/K} \mu) \circ k_A = \pi \quad \text{and} \quad (\pi \times_{G/K} \mu) \circ k_G = \mu.$$

The restriction $\hat{\delta}|_K$ of the dual action $\hat{\delta}$ leaves I_τ invariant, so we get a dual action, denoted $\tilde{\delta}$, of K (not G) on $A \times_{G/K} G$.

If N is another closed normal subgroup of G contained in K , then τ is also a twist for the restricted coaction $(A, G/N, \delta|)$, over the quotient $G/K \cong (G/N)/(K/N)$, so there is a restricted twisted crossed product $A \times_{G/K} G/N$. We have $A \times_{G/K} G/K \cong A$. When Mansfield imprimitivity works for N , we want Mansfield's Hilbert module $Y_{G/N}^G$ to pass to an $A \times_{G/K} G \times_r N$ - $A \times_{G/K} G/N$ imprimitivity bimodule. When $N = K$ is amenable (and δ is reduced), [PR94] shows how to do this: if I_τ and I_τ^K are the twisting ideals of $A \times G$ and $A \times G/K$, respectively, then [PR94] shows $I_\tau \times K = Y_{G/K}^G \text{-Ind } I_\tau^K$, so $Y_{G/K}^G / (Y_{G/K}^G \cdot I_\tau^K)$ gives a Morita equivalence between $(A \times G \times K) / (I_\tau \times K) \cong A \times_{G/K} G \times K$ and $(A \times G/K) / I_\tau^K \cong A$. In the absence of amenability, we run into trouble since we need to identify a quotient of the *reduced* crossed product, so we cannot use universal properties as in [PR94]. We need to know the ideals of $A \times G \times_r N$ and $A \times G/N$ match up suitably. More precisely,

if I_τ^N is the twisting ideal of $A \times G/N$, we need to know

$$(4.1) \quad (A \times G \times_r N) / (Y_{G/N}^G\text{-Ind } I_\tau^N) \cong A \times_{G/K} G \times_r N.$$

Let X be Green's bimodule for inducing representations from $A \times G$ to $A \times G \times N$. X is an $A \times G \times N \times N - A \times G$ imprimitivity bimodule, which we view as a right-Hilbert $A \times G \times_r N - A \times G$ bimodule via the canonical map $j_{A \times G \times_r N}: A \times G \times_r N \rightarrow M(A \times G \times_r N \times N) \cong M(A \times G \times N \times N)$. By the commutativity of the diagram

$$\begin{array}{ccc} A \times G \times N & \xrightarrow{j_{A \times G \times N}} & M(A \times G \times N \times N) \\ \downarrow q & & \downarrow \cong \\ A \times G \times_r N & \xrightarrow{j_{A \times G \times_r N}} & M(A \times G \times_r N \times N), \end{array}$$

the left action of $A \times G \times_r N$ on X is determined by the left actions of $A \times G$ and N ; we will use this when we compute with X in the proof of Theorem 4.1.

Now, since

$$A \times_{G/K} G \times_r N \cong \text{im} (X\text{-Ind}(k_A^G \times k_G)),$$

(4.1) is equivalent to

$$(4.2) \quad Y_{G/N}^G\text{-Ind } I_\tau^N = X\text{-Ind } I_\tau,$$

where I_τ is the twisting ideal of $A \times G$. Since $Y_{G/N}^G$ is an $A \times G \times_r N - A \times G/N$ imprimitivity bimodule, (4.2) is equivalent to

$$I_\tau^N = \widetilde{Y_{G/N}^G\text{-Ind}} (X\text{-Ind } I_\tau),$$

which is the same as

$$(4.3) \quad \ker k_A^{G/N} \times k_G = \ker \widetilde{Y_{G/N}^G\text{-Ind}} (X\text{-Ind}(k_A^G \times k_G)).$$

Assuming for the moment the result of Theorem 4.1, this reduces to

$$(4.4) \quad \ker k_A^{G/N} \times k_G = \ker k_A^G \times k_G|,$$

which becomes

$$\ker k_A^G \times_{G/K} k_G| = \{0\}$$

on passing to representations of the restricted twisted crossed product $A \times_{G/K} G/N$. Hence, in order to ensure that Mansfield imprimitivity passes to the twisted crossed products, we need only the fidelity of

$$(4.5) \quad k_A^G \times_{G/K} k_G|: A \times_{G/K} G/N \rightarrow M(A \times_{G/K} G).$$

Before we investigate this condition further, we should pause to justify the passage from (4.3) to (4.4). This is a consequence of the following result, which is a special case of [KQR95, Theorem 3.1]. Although the proof of that particular result doesn't depend on the results in the present paper, for the reader's peace of mind we give here a complete proof of this special case.

Theorem 4.1. *Let (A, G, δ) be a nondegenerate coaction, and let N be a closed normal subgroup of G such that Mansfield imprimitivity works for N (which is automatic if N is amenable). Then for any representation $\pi \times \mu$ of $A \times G$, the representation*

$$\widetilde{Y_{G/N}^G}\text{-Ind}(X\text{-Ind}(\pi \times \mu))$$

of $A \times G/N$ is unitarily equivalent to $\pi \times \mu|$.

Proof. It is straightforward to check that the map $\pi \times \mu \mapsto \pi \times \mu|$ of $\text{Rep } A \times G$ into $\text{Rep } A \times G/N$ is implemented by viewing $A \times G$ as a right-Hilbert $A \times G/N - A \times G$ bimodule, using the map $j_A \times j_G| : A \times G/N \rightarrow M(A \times G)$. Hence, in order to establish the theorem, it suffices to show that

$$\widetilde{Y_{G/N}^G} \otimes_{A \times G \times_r N} X \cong A \times G$$

as a right-Hilbert $A \times G/N - A \times G$ bimodule. We will actually prove the assertion that

$$Y_{G/N}^G \otimes_{A \times G/N} A \times G \cong X$$

as a right-Hilbert $A \times G \times_r N - A \times G$ bimodule, which is equivalent because $Y_{G/N}^G$ is an imprimitivity bimodule.

Both bimodules in the tensor product are completions of Mansfield's dense subalgebra \mathcal{D} of $A \times G$ for the appropriate inner products. Our isomorphism will be the extension to $Y_{G/N}^G \otimes_{A \times G/N} A \times G$ of the map $\Phi : \mathcal{D} \odot \mathcal{D} \rightarrow C_c(N, \mathcal{D})$ defined by

$$\Phi(x \otimes y)(h) = x \hat{\delta}_h(y).$$

Note that except for the modular function, $\Phi(x \otimes y)$ is just Mansfield's left $C_c(N, \mathcal{D})$ -valued inner product ${}_{A \times G \times_r N} \langle x, y^* \rangle$, and hence does indeed give an element of $C_c(N, \mathcal{D})$. In fact, if we define $f'(n) = f(n) \Delta_N(n)^{-\frac{1}{2}}$, then the map $f \mapsto f'$ is a homeomorphism of $C_c(N, A \times G)$ (with the inductive limit topology) onto itself, which takes $\Phi(\mathcal{D} \odot \mathcal{D})$ to ${}_{A \times G \times_r N} \langle \mathcal{D}, \mathcal{D} \rangle$. This latter set is dense in $C_c(N, A \times G)$ for the inductive limit topology ([Man91, Lemma 25]); it follows that the range of Φ is also inductive limit dense in $C_c(N, A \times G)$, and therefore in X .

It only remains to show that Φ preserves the Hilbert module structure. For the left action of $A \times G \times_r N$, fix $d \in \mathcal{D} \subset A \times G$ and $h, t \in N$. Then:

$$\begin{aligned} d \cdot \Phi(x \otimes y)(h) &= d \Phi(x \otimes y)(h) \\ &= dx \hat{\delta}_h(y) \\ &= \Phi(dx \otimes y) \\ &= \Phi(d \cdot x \otimes y) \end{aligned}$$

and

$$\begin{aligned} t \cdot \Phi(x \otimes y)(h) &= \hat{\delta}_t(\Phi(x \otimes y)(t^{-1}h)) \Delta(t)^{\frac{1}{2}} \\ &= \hat{\delta}_t(x \hat{\delta}_{t^{-1}h}(y)) \Delta(t)^{\frac{1}{2}} \\ &= \hat{\delta}_t(x) \hat{\delta}_h(y) \Delta(t)^{\frac{1}{2}} \\ &= (t \cdot x) \hat{\delta}_h(y) \\ &= \Phi(t \cdot x \otimes y)(h). \end{aligned}$$

For the right action of $A \times G$, fix $d \in \mathcal{D} \subset A \times G$ and $h \in N$. Then:

$$\begin{aligned} \Phi(x \otimes y) \cdot d(h) &= \Phi(x \otimes y)(h) \hat{\delta}_h(d) \\ &= x \hat{\delta}_h(yd) \\ &= \Phi(x \otimes (yd))(h) \\ &= \Phi(x \otimes y \cdot d)(h). \end{aligned}$$

For the right $A \times G$ -valued inner product, compute:

$$\begin{aligned} &\langle \Phi(x \otimes y), \Phi(z \otimes w) \rangle_{A \times G} \\ &= \int_N \hat{\delta}_h(\Phi(x \otimes y)(h^{-1})^* \Phi(z \otimes w)(h^{-1})) dh \\ &= \int_N \hat{\delta}_h((x \hat{\delta}_{h^{-1}}(y))^* z \hat{\delta}_{h^{-1}}(w)) dh \\ &= y^* \int_N \hat{\delta}_h(x^* z) dh w \\ &= y^* \langle x, z \rangle_{A \times G/N} w \\ &= y^* \left(\langle x, z \rangle_{A \times G/N} \cdot w \right) \\ &= \left\langle y, \langle x, z \rangle_{A \times G/N} \cdot w \right\rangle_{A \times G} \\ &= \langle x \otimes y, z \otimes w \rangle_{A \times G}. \end{aligned}$$

This completes the proof of the theorem. \square

Of course, the discussion preceding Theorem 4.1 was predicated on Mansfield imprimitivity working for N and δ ; however, $k_A^G \times_{G/K} k_G$ is certainly well-defined for any twisted coaction. It will turn out (Theorem 4.3) that for a twisted coaction $(A, G, G/K, \delta, \tau)$ and a closed normal subgroup N of G contained in K , this map is faithful exactly when $j_A^G \times j_G$ is. In order to see this, we will need the following variation on [QR95, Lemma 3.5]. Recall that for an action (B, K, α) of a closed subgroup K of a locally compact group G , the *induced algebra* $\text{Ind}_K^G B$ is the C^* -algebra of continuous maps $f: G \rightarrow B$ such that $f(sk) = \alpha_{k^{-1}}(f(s))$ for all $s \in G, k \in K$, and such that the map $sK \mapsto \|f(s)\|$ vanishes at infinity on G/K .

Lemma 4.2. *Let (B, K, α) and (C, K, β) be actions of a closed subgroup K of a locally compact group G , and let $\varphi: B \rightarrow M(C)$ be a K -equivariant nondegenerate homomorphism. Then φ is faithful if and only if the induced homomorphism $\text{Ind } \varphi: \text{Ind}_K^G B \rightarrow M(\text{Ind}_K^G C)$ is, where $\text{Ind } \varphi$ is defined by*

$$\text{Ind } \varphi(f)(s) = \varphi(f(s)).$$

Proof. If φ has nontrivial kernel, then $\ker \varphi$ is a nonzero K -invariant ideal of B , so $\text{Ind}(\ker \varphi)$ is a nonzero subset of $\ker(\text{Ind } \varphi)$: for $f \in \text{Ind}(\ker \varphi)$ and $s \in G$,

$$\text{Ind } \varphi(f)(s) = \varphi(f(s)) = 0.$$

Conversely, if f is a nonzero element of $\ker(\text{Ind } \varphi)$, then $f(G)$ is a nonzero subset of $\ker \varphi$. \square

Theorem 4.3. *Let $(A, G, G/K, \delta, \tau)$ be a twisted coaction, and let N be a closed normal subgroup of G contained in K . Then*

$$j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$$

is faithful if and only if

$$k_A \times_{G/K} k_G|: A \times_{G/K} G/N \rightarrow M(A \times_{G/K} G)$$

is faithful.

Proof. Note that the diagram

$$\begin{array}{ccc} A \times G/N & \xrightarrow{j_A \times j_G|} & M(A \times G) \\ k_A \times_{G/N} \downarrow & & \downarrow k_A \times_{G/K} \\ A \times_{G/K} G/N & \xrightarrow{k_A \times_{G/K} k_G|} & M(A \times_{G/K} G) \end{array}$$

commutes.

By [QR95, Theorem 4.4], the formula

$$\Phi(x)(s) = (k_A \times k_G) \circ \hat{\delta}_{s^{-1}}(x) \quad \text{for } x \in A \times G, s \in G$$

defines an isomorphism $\Phi: A \times G \rightarrow \text{Ind}_K^G A \times_{G/K} G$, and similarly

$$A \times G/N \cong \text{Ind}_{K/N}^{G/N} A \times_{G/K} G/N.$$

We need to do everything in terms of K and G rather than their quotients by N . Define $\alpha: G \rightarrow \text{Aut } A \times G/N$ by

$$\alpha_s = (\delta|)^{\sim}_{sN}.$$

Then

$$(j_A \times j_G|) \circ \alpha_s = \hat{\delta}_s \circ (j_A \times j_G|) \quad \text{for } s \in G.$$

The twisting ideal I_τ^N of $A \times G/N$ is invariant under the restriction $\alpha|_K$, so there is a unique action $\tilde{\alpha}$ of K on $A \times_{G/K} G/N$ such that

$$\tilde{\alpha}_s \circ (k_A \times k_{G/N}) = (k_A \times k_{G/N}) \circ \alpha_s \quad \text{for } s \in K.$$

Of course, $\tilde{\alpha}_s = (\delta|)^{\sim}_{sN}$.

An easy calculation shows the formula

$$\Psi(f)(s) = f(sN) \quad \text{for } s \in G$$

gives an isomorphism

$$\Psi: \text{Ind}_{K/N}^{G/N} A \times_{G/K} G/N \rightarrow \text{Ind}_K^G A \times_{G/K} G/N.$$

Combining this with [QR95, Theorem 4.4], we can define an isomorphism

$$\Phi_N: A \times G/N \rightarrow \text{Ind}_K^G A \times_{G/K} G/N$$

by

$$\Phi_N(x)(s) = (k_A \times k_{G/N}) \circ \alpha_{s^{-1}}(x).$$

The following calculation shows

$$k_A \times_{G/K} k_G|: A \times_{G/K} G/N \rightarrow M(A \times_{G/K} G)$$

is K -equivariant: for $s \in K$

$$\begin{aligned} & (k_A \times_{G/K} k_G|) \circ \tilde{\alpha}_s \circ (k_A \times k_{G/N}) \\ &= (k_A \times_{G/K} k_G|) \circ (k_A \times k_{G/N}) \circ \alpha_s \\ &= (k_A \times k_G) \circ (j_A \times j_G|) \circ \alpha_s \\ &= (k_A \times k_G) \circ \hat{\delta}_s \circ (j_A \times j_G|) \\ &= \tilde{\delta}_s \circ (k_A \times k_G) \circ (j_A \times j_G|) \\ &= \tilde{\delta}_s \circ (k_A \times_{G/K} k_G|) \circ (k_A \times k_{G/N}). \end{aligned}$$

We next show the nondegenerate homomorphism

$$\Phi \circ (j_A \times j_G|) \circ \Phi_N^{-1}: \text{Ind}_K^G A \times_{G/K} G/N \rightarrow M(\text{Ind}_K^G A \times_{G/K} G)$$

is induced from

$$k_A \times_{G/K} k_G|: A \times_{G/K} G/N \rightarrow M(A \times_{G/K} G)$$

in the sense of Lemma 4.2, and then the lemma will yield the present theorem. For $f \in \text{Ind}_K^G A \times_{G/K} G/N$, $s \in G$,

$$\begin{aligned} & \Phi \circ (j_A \times j_G|) \circ \Phi_N^{-1}(f)(s) \\ &= (k_A \times k_G) \circ \hat{\delta}_{s^{-1}} \circ (j_A \times j_G|) \circ \Phi_N^{-1}(f) \\ &= (k_A \times k_G) \circ (j_A \times j_G|) \circ \alpha_{s^{-1}} \circ \Phi_N^{-1}(f) \\ &= (k_A \times_{G/K} k_G|) \circ (k_A \times k_{G/N}) \circ \alpha_{s^{-1}} \circ \Phi_N^{-1}(f) \\ &= (k_A \times_{G/K} k_G|)(\Phi_N(\Phi_N^{-1}(f))(s)) \\ &= (k_A \times_{G/K} k_G|)(f(s)). \end{aligned}$$

□

Now the discussion encompassing Equations (4.1)–(4.5), together with Theorem 4.3, gives the following extension of [PR89, Theorem 4.1]. In the case where N is amenable, this result is hinted at in the discussion preceding [ER, Theorem 4.7].

Theorem 4.4. *Let $(A, G, G/K, \delta, \tau)$ be a nondegenerate twisted coaction, and let N a closed normal subgroup of G contained in K such that Mansfield imprimitivity works for N and δ (which is automatic if N is amenable). Then the quotient $Z_{G/N}^G = Y_{G/N}^G / (Y_{G/N}^G \cdot I_\tau^N)$ is an $A \times_{G/K} G \times_\tau N$ – $A \times_{G/K} G/N$ imprimitivity bimodule.*

5. SUBGROUPS, MORITA EQUIVALENCE, INFLATION, STABILIZATION

In this section, we will show that Mansfield imprimitivity is compatible with many of the standard coaction constructions. As a warmup, we show that Mansfield imprimitivity passes to subgroups. First we need the following variation on [QR95, Corollary 4.10]. Sadly, the hypothesis in [QR95] seems to be deficient: equivariance must be imposed on the integrated form of the pair (π, μ) , rather than just μ .

Proposition 5.1. *Let $(A, G, G/K, \delta, \tau)$ be a twisted coaction and (π, μ) a covariant representation preserving the twist. Then the representation $\pi \times_{G/K} \mu$ of $A \times_{G/K} G$ is faithful if and only if $\ker \pi \subset \ker j_A$ and*

there is an action α of K on $\text{im } \pi \times \mu$ such that

$$\alpha_s \circ (\pi \times_{G/K} \mu) = (\pi \times_{G/K} \mu) \circ \tilde{\delta}_s \quad \text{for } s \in K.$$

Proof. Replace the twisted coaction by its reduction [Qui95, Corollary 3.8], [Rae92, Theorem 4.1]; then j_A is faithful, and [QR95, Corollary 4.10] gives the proposition. \square

Theorem 5.2. *Let (A, G, δ) be a nondegenerate coaction and $N \subset H$ closed normal subgroups of G . If Mansfield imprimitivity works for H , then it also works for N .*

Proof. The diagram

$$\begin{array}{ccc} M(C_0(G)) & \xleftarrow{p_N} & M(C_0(G/N)) \\ p_H \uparrow & & \uparrow p_{H/N} \\ C_0(G/H) & \xleftarrow[\gamma]{\cong} & C_0((G/N)/(H/N)) \end{array}$$

commutes, where γ is the natural isomorphism, and this gives us a commutative diagram

$$\begin{array}{ccc} M(A \times G) & \xleftarrow{j_A \times (j_G \circ p_N)} & M(A \times G/N) \\ j_A \times (j_G \circ p_H) \uparrow & & \uparrow j_A \circ (j_{G/N} \circ p_{H/N}) \\ A \times G/H & \xleftarrow[j_A \times (j_{G/H} \circ \gamma)]{\cong} & A \times (G/N)/(H/N). \end{array}$$

Assuming $j_A \times (j_G \circ p_H)$ is faithful, so is $j_A \times (j_{G/N} \circ p_{H/N})$. So, [PR89, Theorem 3.1] applies, giving a decomposition isomorphism

$$\theta: A \times G/N \xrightarrow{\cong} A \times (G/N)/(H/N) \times_{(G/N)/(H/N)} G/N,$$

hence a homomorphism

$$\begin{aligned} \sigma &:= (j_A \times (j_G \circ p_N)) \circ \theta^{-1} \\ &= (j_A \times (j_G \circ p_N \circ p_{H/N})) \times_{(G/N)/(H/N)} (j_G \circ p_N) \end{aligned}$$

of $A \times (G/N)/(H/N) \times_{(G/N)/(H/N)} G/N$. We aim to apply the preceding proposition to show σ is faithful. Since the decomposition coaction of G/N on $A \times (G/N)/(H/N)$ is normal, we must show the homomorphism $j_A \times (j_G \circ p_N \circ p_{H/N})$ of $A \times (G/N)/(H/N)$ is faithful and there is an action α of H/N on $\text{im } \sigma$ such that

$$\alpha_{hN} \circ j_G \circ p_N(f) = j_G \circ p_N(hN \cdot f) \quad \text{for } h \in H, f \in C_0(G/N).$$

The first follows from

$$j_A \times (j_G \circ p_N \circ p_{H/N}) = (j_A \times (j_G \circ p_H)) \circ (j_A \times (j_{G/H} \circ \gamma))$$

and fidelity of $j_A \times (j_G \circ p_H)$. For the second, since the restriction $\hat{\delta}|_N$ is trivial on $\text{im } \sigma = \text{im } j_A \times (j_G \circ p_N)$, there is a unique action α of H/N on $\text{im } \sigma$ such that

$$\alpha_{hN} = \hat{\delta}_h \quad \text{for } h \in H.$$

For $f \in C_0(G/N)$,

$$\begin{aligned} \alpha_{hN} \circ j_G \circ p_N(f) &= \hat{\delta}_h \circ j_G(p_N(f)) \\ &= j_G(h \cdot p_N(f)) \\ &= j_G \circ p_N(hN \cdot f). \end{aligned}$$

□

We now show Mansfield imprimitivity is preserved by Morita equivalence of coactions. Our conventions are those of [Ng95].

Theorem 5.3. *Let (A, G, δ_A) and (B, G, δ_B) be Morita equivalent coactions with one (hence both) nondegenerate, and let N be a closed normal subgroup of G . Then Mansfield imprimitivity works for N and δ_A if and only if it works for N and δ_B .*

Proof. Let (X, δ_X) be a Morita equivalence for (A, G, δ_A) and (B, G, δ_B) ; so $\delta = (\delta_A, \delta_X, \delta_B)$ is a coaction of G on ${}_A X_B$. Nondegeneracy of both coactions follows from nondegeneracy of either one by Proposition 2.3.

As usual, let $(A \times G, j_A^G, j_G^A)$ and $(B \times G, j_B^G, j_G^B)$ denote the crossed products for (A, G, δ_A) and (B, G, δ_B) , respectively. By the uniqueness of the imprimitivity bimodule crossed product ([Ng95, Remark 3.6(c)]), we can suppose that the crossed product for $({}_A X_B, G, \delta)$ is of the form

$$({}_A \times G(X \times G)_{B \times G}, j_A^G, j_X^G, j_B^G, j_G^A, j_G^B)$$

for some linear map $j_X^G: X \rightarrow M(X \times G)$ (cf. [Ng95, Definition 3.5(b)]).

The conscientious reader will check that

$$(j_A^G, j_X^G, j_B^G, j_G^A, j_G^B)$$

is a covariant representation of the restricted imprimitivity bimodule coaction $({}_A X_B, G/N, \delta)$ in the sense of [Ng95, Definition 3.5(a)]. It then follows that there is a unique imprimitivity bimodule representation $(\varphi_{A \times G/N}, \varphi_{X \times G/N}, \varphi_{B \times G/N})$ of ${}_A \times G/N(X \times G/N)_{B \times G/N}$ such that

$$\begin{aligned} &(\varphi_{A \times G/N} \circ j_A^{G/N}, \varphi_{X \times G/N} \circ j_X^{G/N}, \varphi_{B \times G/N} \circ j_B^{G/N}, \\ &\varphi_{A \times G/N} \circ j_G^A, \varphi_{B \times G/N} \circ j_G^B) \\ &= (j_A^G, j_X^G, j_B^G, j_G^A, j_G^B). \end{aligned}$$

In particular, $\varphi_{A \times G/N}$ is a representation of $A \times G/N$ satisfying

$$\varphi_{A \times G/N} \circ j_A^{G/N} = j_A^G$$

and

$$\varphi_{A \times G/N} \circ j_{G/N}^A = j_G^A|,$$

so we have $\varphi_{A \times G/N} = j_A^G \times j_G^A|$. By the same token, we have $\varphi_{B \times G/N} = j_B^G \times j_G^B|$. Now [ER, Lemma 2.7] tells us the ideals $\ker(j_A^G \times j_G^A|)$ of $A \times G/N$ and $\ker(j_B^G \times j_G^B|)$ of $B \times G/N$ correspond via $X \times G/N$; in particular, $j_A^G \times j_G^A|$ is faithful if and only if $j_B^G \times j_G^B|$ is. This establishes the theorem. \square

Next, we turn to inflation. Recall that if $(A, G, \inf \epsilon)$ is inflated from (A, K, ϵ) , then $\inf \epsilon$ is trivially twisted over G/K by $f \mapsto f(e)1$, and [PR89, Example 2.14] gives a natural isomorphism

$$A \times_{G/K} G \cong A \times K.$$

Theorem 5.4. *Let $N \subset K$ be closed normal subgroups of G , and let (A, K, ϵ) be a coaction such that either ϵ or $\inf \epsilon$ (hence the other) is nondegenerate. Then Mansfield imprimitivity works for N and ϵ if and only if it works for N and $\inf \epsilon$.*

Proof. Nondegeneracy of both coactions follows from nondegeneracy of either one by Proposition 2.1.

Consider the twisted inflated coaction $(A, G, G/K, \inf \epsilon, 1)$. The diagram

$$\begin{array}{ccc} A \times K/N & \xrightarrow{j_A \times j_K|} & M(A \times K) \\ \cong \downarrow & & \downarrow \cong \\ A \times_{G/K} G/N & \xrightarrow{k_A \times_{G/K} k_G|} & M(A \times_{G/K} G) \end{array}$$

commutes, so $j_A \times j_K|$ is faithful if and only if $k_A \times_{G/K} k_G|$ is. Theorem 4.3 tells us this latter is equivalent to fidelity of $j_A \times j_G|: A \times G/N \rightarrow M(A \times G)$, and this is enough to finish the proof. \square

Finally, we chain the above results together to show that Mansfield imprimitivity is compatible with the stabilization trick of [ER]. Let $(A, G, G/K, \delta, \tau)$ be a nondegenerate twisted coaction such that Mansfield imprimitivity works for δ and K itself. In light of Lemma 3.6, this is equivalent to δ being normal. We have a dual action $\tilde{\delta}$ of K on the twisted crossed product $A \times_{G/K} G$, hence a double dual coaction $\widehat{\tilde{\delta}}$ of K on the full crossed product $A \times_{G/K} G \times K$. The normalization $(\widehat{\tilde{\delta}})^n$ of this coaction is on the reduced crossed product $A \times_{G/K} G \times_r K$ [Rae92, Proposition 3.2 (1)], [Qui95, Propositions 3.6 and 3.7]. By [ER, Theorem 3.1], the twisted coaction $(A, G, G/K, \delta, \tau)$ is Morita equivalent to the inflated twisted coaction $(A \times_{G/K} G \times_r K, G, G/K, \inf(\widehat{\tilde{\delta}})^n, 1)$. Even

though [ER] uses reduced coactions and requires N to be amenable, their arguments carry over to our setting since we assume that Mansfield imprimitivity works. The necessary adjustments are fairly obvious, such as replacing [ER, Lemma 3.11] with

$$\hat{\delta}_n \otimes \iota(j_G \otimes \iota(w_G)) = j_G \otimes \iota(w_G)(1 \otimes n) \quad \text{for } n \in N.$$

We do not need the twisting subgroup K to be amenable, since we use full coactions—see [QR95, Section 7].

Theorem 5.5. *Let $(A, G, G/K, \delta, \tau)$ be a nondegenerate normal twisted coaction. Then the stabilized coaction $(A \times_{G/K} G \times_r K, K, (\hat{\delta})^n)$ is also nondegenerate. Furthermore, if N is a closed normal subgroup of G contained in K , then Mansfield imprimitivity works for N and $(\hat{\delta})^n$.*

Proof. By [ER, Theorem 3.1], (A, G, δ) is Morita equivalent to the inflated coaction $(A \times_{G/K} G \times_r K, G, \inf(\hat{\delta})^n)$; thus $\inf(\hat{\delta})^n$ is nondegenerate by Proposition 2.3, and hence also $(\hat{\delta})^n$ by Proposition 2.1.

Now let N be a closed normal subgroup of G contained in K . Since δ is normal, Mansfield imprimitivity works for N and δ (Lemma 3.2), and hence it also works for N and the Morita equivalent coaction $\inf(\hat{\delta})^n$ (Theorem 5.3). It follows that Mansfield imprimitivity works for N and the deflated coaction $(\hat{\delta})^n$, by Theorem 5.4. \square

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